Algebraic Rainich conditions for the tensor V

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Abstract

Algebraic conditions on the Ricci tensor in the Rainich-Misner-Wheeler unified field theory are known as the Rainich conditions. Penrose and more recently Bergqvist and Lankinen made an analogy from the Ricci tensor to the Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$, a certain fourth rank tensor quadratic in the Weyl curvature, which also satisfies algebraic Rainich-like conditions. However, we found that not only does the tensor $B_{\alpha\beta\mu\nu}$ fulfill these conditions, but so also does our recently proposed tensor $V_{\alpha\beta\mu\nu}$, which has many of the desirable properties of $B_{\alpha\beta\mu\nu}$. For the quasilocal small sphere limit restriction, we found that there are only two fourth rank tensors $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ which form a basis for good energy expressions. Both of them have the completely trace free and causal properties, these two form necessary and sufficient conditions. Surprisingly either completely traceless or causal is enough to fulfill the algebraic Rainich conditions. Furthermore, relaxing the quasilocal restriction and considering the general fourth rank tensor, we found two remarkable results: (i) without any symmetry requirement, the algebraic Rainich conditions only require totally trace free; (ii) with a symmetry requirement, we recovered the same result as in the quasilocal small sphere limit.

1 Introduction

In 1925 Rainich proposed a unified field theory for source-free electromagnetism and gravitation [1]. Misner and Wheeler [2] 32 years later proposed a geometrically unified theory, based on the Rainich idea, now called the Rainich-Misner-Wheeler theory. The necessary conditions are called the Rainich conditions. The algebraic Rainich conditions refer to the Ricci tensor, but this tensor can be replaced by other tensors. Penrose [3] and more recently Bergqvist and Lankinen [4, 5] made an analogy from the Ricci tensor to the Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$, a certain fourth rank tensor quadratic in the Weyl curvature, which also satisfies algebraic Rainich-like conditions. We found that not only does the tensor $B_{\alpha\beta\mu\nu}$ fulfill these conditions, but so also does our recently proposed tensor $V_{\alpha\beta\mu\nu}$, which has many of the desirable properties of $B_{\alpha\beta\mu\nu}$.

The Bel-Robinson tensor possesses many nice properties; it is completely symmetric, totally trace-free and divergence-free. It also satisfies the dominant energy condition [6],

$$B_{\alpha\beta\mu\nu}u^{\alpha}v^{\beta}w^{\mu}z^{\nu}\geq 0, \tag{1}$$

for any timelike unit normal vectors u, v, w and z. An unique alternative tensor $V_{\alpha\beta\mu\nu}$ was proposed recently which gives the same non-negative gravitational energy

density in the small sphere limit [7]

$$V_{\alpha\beta\mu\nu} := S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu} \equiv B_{\alpha\beta\mu\nu} + W_{\alpha\beta\mu\nu}, \tag{2}$$

where $B_{\alpha\beta\mu\nu}$, $S_{\alpha\beta\mu\nu}$, $K_{\alpha\beta\mu\nu}$ and $W_{\alpha\beta\mu\nu}$ are defined in section 2. While $V_{\alpha\beta\mu\nu}$ [7, 8] does not have the completely symmetric property, it does fulfill the totally trace free property and satisfies the weak energy condition,

$$V_{\alpha\beta\mu\nu} u^\alpha u^\beta u^\mu u^\nu \equiv B_{\alpha\beta\mu\nu} u^\alpha u^\beta u^\mu u^\nu \geq 0. \tag{3}$$

The algebraic Rainich conditions [1, 9] are

$$R_{\alpha\lambda} R_\beta{}^\lambda = \frac{1}{4} g_{\alpha\beta} R_{\rho\lambda} R^{\rho\lambda}, \qquad R^\lambda{}_\lambda = 0, \qquad R_{\alpha\beta} u^\alpha u^\beta \geq 0, \tag{4}$$

where $R_{\alpha\beta}$ is the second rank Ricci tensor. Turning to higher rank, one can use the Bel-Robinson tensor which is the first fourth rank tensor people recognized that satisfies these algebraic Rainich conditions [3, 4]. It is known that the Ricci tensor is symmetric; if we make an analogy from second rank to fourth rank, the completely symmetric property need not be required. Therefore, as far as the quasilocal small sphere limit is concerned, we found the unique alternative fourth rank tensor $V_{\alpha\beta\mu\nu}$ which satisfies the algebraic Rainich conditions.

Interestingly, we discovered that $X_{\alpha\lambda\sigma\tau} Y_\beta{}^{\lambda\sigma\tau} = \frac{1}{4} g_{\alpha\beta} X_{\rho\lambda\sigma\tau} Y^{\rho\lambda\sigma\tau}$, where $X_{\alpha\beta\mu\nu}$ and $Y_{\alpha\beta\mu\nu}$ are any quadratic in Riemann curvature tensors. This indicates that this is an identity (i.e., not restricted to the quasilocal small sphere limit) which means it is no longer a condition. Therefore the algebraic Rainich conditions for fourth rank leave two conditions, not the expected three.

Under the quasilocal small sphere limit restriction, we found that there are only two fourth rank tensors $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ forming a basis for good expressions. Both of them have the completely trace free and causal properties, we found that these two properties form necessary and sufficient conditions. In other words, either completely traceless or causal can fulfill the algebraic Rainich conditions.

Furthermore, relaxing the quasilocal restriction and considering the general fourth rank tensor, we found two important results. One is without any symmetry requirement: we found that the algebraic condition only requires completely traceless. The other is imposing some certain symmetry: we recovered, as expected, the same result as in the quasilocal small sphere limit.

2 Technical background

The Bel-Robinson tensor [10] was proposed in 1958 as a certain quadratic combination of the Weyl tensor:

$$\begin{aligned} B_{\alpha\beta\mu\nu} &:= R_{\alpha\lambda\mu\sigma} R_\beta{}^\lambda{}_\nu{}^\sigma + *R_{\alpha\lambda\mu\sigma} * R_\beta{}^\lambda{}_\nu{}^\sigma \\ &= R_{\alpha\lambda\mu\sigma} R_\beta{}^\lambda{}_\nu{}^\sigma + R_{\alpha\lambda\nu\sigma} R_\beta{}^\lambda{}_\mu{}^\sigma - \frac{1}{2} g_{\alpha\beta} R_{\mu\lambda\sigma\tau} R_\nu{}^{\lambda\sigma\tau}, \end{aligned} \tag{5}$$

where $*R_{\alpha\lambda\sigma\tau}$ is the dual of $R_{\alpha\lambda\sigma\tau}$. One place where the Bel-Robinson tensor naturally shows up is in the expressions for gravitational energy in a small region. There are three fundamental tensors that commonly occur in the gravitational pseudotensor expressions [11, 12] in vacuum:

$$
\begin{aligned}
B_{\alpha\beta\mu\nu} &:= R_{\alpha\lambda\mu\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma} + R_{\alpha\lambda\nu\sigma}R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma} - \frac{1}{8}g_{\alpha\beta}g_{\mu\nu}\mathbf{R}^2, && (6)\\
S_{\alpha\beta\mu\nu} &:= R_{\alpha\mu\lambda\sigma}R_{\beta\nu}{}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma}R_{\beta\mu}{}^{\lambda\sigma} + \frac{1}{4}g_{\alpha\beta}g_{\mu\nu}\mathbf{R}^2, && (7)\\
K_{\alpha\beta\mu\nu} &:= R_{\alpha\lambda\beta\sigma}R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma} + R_{\alpha\lambda\beta\sigma}R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma} - \frac{3}{8}g_{\alpha\beta}g_{\mu\nu}\mathbf{R}^2. && (8)
\end{aligned}
$$

where $\mathbf{R}^2 = R_{\rho\tau\xi\kappa}R^{\rho\tau\xi\kappa}$ and we have rewritten (5) by substituting the well known identity [13] in vacuum

$$
R_{\mu\lambda\sigma\tau}R_{\nu}{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\mu\nu}\mathbf{R}^2. \tag{9}
$$

On the other hand, we also introduced another tensor $W_{\alpha\beta\mu\nu}$ which gives an alternative representation of $V_{\alpha\beta\mu\nu}$ as denoted in (2)

$$
W_{\alpha\beta\mu\nu} := \frac{3}{2}S_{\alpha\beta\mu\nu} - \frac{5}{8}g_{\alpha\beta}g_{\mu\nu}\mathbf{R}^2 + \frac{1}{8}(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu})\mathbf{R}^2, \tag{10}
$$

The analog of the electric part E_{ab} and magnetic part H_{ab} are defined in terms of the Weyl tensor [14] as follows

$$
E_{ab} := C_{a0b0}, \quad H_{ab} := *C_{a0b0}, \quad a, b = 1, 2, 3. \tag{11}
$$

The fundamental property of a tensor is that if it vanishes in one frame, then it must vanish in any other frame. This is an elementary property for a tensor, however, remarkably it provides an easy and efficient way for the verification of identities. For instance, we have checked using orthonormal frames that the identity in (9) is true.

Here are some properties of $S_{\alpha\beta\mu\nu}$, $K_{\alpha\beta\mu\nu}$, $W_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ that we found:

$$
\begin{aligned}
S_{\alpha\beta\mu\nu} &\equiv S_{(\alpha\beta)(\mu\nu)} \equiv S_{(\mu\nu)(\alpha\beta)}, && (12)\\
S_{\alpha\beta\mu}{}^{\mu} &\equiv \frac{1}{4}g_{\alpha\beta}S_{\rho}{}^{\rho}{}_{\mu}{}^{\mu} \equiv \frac{3}{2}g_{\alpha\beta}\mathbf{R}^2, \quad S_{\alpha\mu\beta}{}^{\mu} \equiv \frac{1}{4}g_{\alpha\beta}S_{\rho\mu}{}^{\rho\mu} \equiv 0, && (13)\\
K_{\alpha\beta\mu\nu} &\equiv K_{(\alpha\beta)(\mu\nu)} \equiv K_{(\mu\nu)(\alpha\beta)}, && (14)\\
K_{\alpha\beta\mu}{}^{\mu} &\equiv \frac{1}{4}g_{\alpha\beta}K_{\rho}{}^{\rho}{}_{\mu}{}^{\mu} \equiv -\frac{3}{2}g_{\alpha\beta}\mathbf{R}^2, \quad K_{\alpha\mu\beta}{}^{\mu} \equiv \frac{1}{4}g_{\alpha\beta}K_{\rho\mu}{}^{\rho\mu} \equiv 0, && (15)\\
W_{\alpha\beta\mu\nu} &\equiv W_{(\alpha\beta)(\mu\nu)} \equiv W_{(\mu\nu)(\alpha\beta)}, && (16)\\
W_{\alpha\beta\mu}{}^{\mu} &\equiv \frac{1}{4}g_{\alpha\beta}W_{\rho}{}^{\rho}{}_{\mu}{}^{\mu} \equiv 0, \quad W_{\alpha\mu\beta}{}^{\mu} \equiv \frac{1}{4}g_{\alpha\beta}W_{\rho\mu}{}^{\rho\mu} \equiv 0, && (17)\\
V_{\alpha\beta\mu\nu} &\equiv V_{(\alpha\beta)(\mu\nu)} \equiv V_{(\mu\nu)(\alpha\beta)}, && (18)\\
V_{\alpha\beta\mu}{}^{\mu} &\equiv \frac{1}{4}g_{\alpha\beta}V_{\rho}{}^{\rho}{}_{\mu}{}^{\mu} \equiv 0, \quad V_{\alpha\mu\beta}{}^{\mu} \equiv \frac{1}{4}g_{\alpha\beta}V_{\rho\mu}{}^{\rho\mu} \equiv 0. && (19)
\end{aligned}
$$

Note that, unlike the Bel-Robinson tensor, both $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$ are neither totally symmetric nor totally trace free [8].

3 Quadratic tensor identities for B, S, K, V

Owing to the equivalence principle, gravitational energy cannot be detected at a point. Therefore we use quasilocal methods (including pseudotensors). Dealing with the quasilocal small sphere limit approximation, consider all the possible combinations of the small region energy-momentum density in vacuum; the general expression is [15]

$$2\kappa\, t_{\alpha}{}^{\beta} = 2G_{\alpha}{}^{\beta} + (a_1 \tilde{B}_{\alpha}{}^{\beta}{}_{\mu\nu} + a_2 \tilde{S}_{\alpha}{}^{\beta}{}_{\mu\nu} + a_3 \tilde{K}_{\alpha}{}^{\beta}{}_{\mu\nu} + a_4 \tilde{T}_{\alpha}{}^{\beta}{}_{\mu\nu}) x^{\mu} x^{\nu} + \mathcal{O}(\mathrm{Ricci}, x) + \mathcal{O}(x^3), \tag{20}$$

where $\kappa = 8\pi G/c^4$ (here we take units such that $c = 1$ for simplicity) and a_1 to a_4 are real numbers. For the quadratic curvature tensors in (20), there are four independent basis [7, 16] expressions with certain symmetries which we used:

$$\tilde{B}_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma} R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma} = B_{\alpha\beta\mu\nu} + \frac{1}{8} g_{\alpha\beta} g_{\mu\nu} \mathbf{R}^2, \tag{21}$$

$$\tilde{S}_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\sigma} R_{\beta\nu}{}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma} R_{\beta\mu}{}^{\lambda\sigma} = S_{\alpha\beta\mu\nu} - \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} \mathbf{R}^2, \tag{22}$$

$$\tilde{K}_{\alpha\beta\mu\nu} := R_{\alpha\lambda\beta\sigma} R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma} + R_{\alpha\lambda\beta\sigma} R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma} = K_{\alpha\beta\mu\nu} + \frac{3}{8} g_{\alpha\beta} g_{\mu\nu} \mathbf{R}^2, \tag{23}$$

$$\tilde{T}_{\alpha\beta\mu\nu} := -\frac{1}{8} g_{\alpha\beta} g_{\mu\nu} \mathbf{R}^2. \tag{24}$$

Note that none of these four tensors has the completely symmetric property, e.g., $\tilde{B}_{0011} \neq \tilde{B}_{0101}$ in general. However, all of them do have certain symmetries, precisely $\tilde{X}_{\alpha\beta\mu\nu} = \tilde{X}_{(\alpha\beta)(\mu\nu)} = \tilde{X}_{(\mu\nu)(\alpha\beta)}$. Although there exists some other tensors different from $\tilde{B}_{\alpha\beta\mu\nu}$, $\tilde{S}_{\alpha\beta\mu\nu}$, $\tilde{K}_{\alpha\beta\mu\nu}$ and $\tilde{T}_{\alpha\beta\mu\nu}$, they are just linear combinations of these four. For instance [16]

$$\tilde{T}_{\alpha\mu\beta\nu} + \tilde{T}_{\alpha\nu\beta\mu} \equiv \tilde{B}_{\alpha\beta\mu\nu} + \frac{1}{2} \tilde{S}_{\alpha\beta\mu\nu} - \tilde{K}_{\alpha\beta\mu\nu} + 2\tilde{T}_{\alpha\beta\mu\nu}. \tag{25}$$

The above identity can be obtained by making use of the completely symmetric property of the Bel-Robinson tensor. Using (25), we can rewrite the Bel-Robinson tensor in a different representation [16]:

$$B_{\alpha\beta\mu\nu} \equiv -\frac{1}{2} S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu} + \frac{5}{8} g_{\alpha\beta} g_{\mu\nu} \mathbf{R}^2 - \frac{1}{8} (g_{\alpha\mu} g_{\beta\nu} + g_{\alpha\nu} g_{\beta\mu}) \mathbf{R}^2. \tag{26}$$

There is a known formula for the quadratic Bel-Robinson tensor [3]

$$B_{\alpha\lambda\sigma\tau} B_{\beta}{}^{\lambda\sigma\tau} \equiv \frac{1}{4} g_{\alpha\beta} B_{\rho\lambda\sigma\tau} B^{\rho\lambda\sigma\tau}, \tag{27}$$

which was given by Penrose using spinor methods [3]. We have verified this identity using orthonormal frames (for details see (83) below), moreover, using the same method, we found the following identity

$$S_{\alpha\lambda\sigma\tau} S_{\beta}{}^{\lambda\sigma\tau} \equiv \frac{1}{4} g_{\alpha\beta} S_{\rho\lambda\sigma\tau} S^{\rho\lambda\sigma\tau}. \tag{28}$$

This is a milestone for verifying other quadratic identities (e.g., $K_{\alpha\lambda\sigma\tau}K_\beta{}^{\lambda\sigma\tau}$) in an easier way. In other words, one can use the same method in orthonormal frames to verify all the possible combinations, but it would take much unnecessary work. Instead we used simple algebra substitution. Remember that $B_{\alpha\beta\mu\nu}$ is completely symmetric and trace-free; making use of (7) and (26), we found

$$\begin{aligned} 0 &= S_{\alpha\lambda\sigma\tau}B_\beta{}^{\lambda\sigma\tau} \equiv -\frac{1}{2}S_{\alpha\lambda\sigma\tau}S_\beta{}^{\lambda\sigma\tau} + S_{\alpha\lambda\sigma\tau}K_\beta{}^{\lambda\sigma\tau} + \frac{15}{16}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2, \quad (29)\\ 0 &= S_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} \equiv -\frac{1}{2}S_{\rho\lambda\sigma\tau}S^{\rho\lambda\sigma\tau} + S_{\rho\lambda\sigma\tau}K^{\rho\lambda\sigma\tau} + \frac{15}{4}\mathbf{R}^2\mathbf{R}^2. \quad (30)\end{aligned}$$

Rewrite (29) and (30) as follows:

$$\begin{aligned} S_{\alpha\lambda\sigma\tau}K_\beta{}^{\lambda\sigma\tau} &\equiv \frac{1}{2}S_{\alpha\lambda\sigma\tau}S_\beta{}^{\lambda\sigma\tau} - \frac{15}{16}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2, \quad (31)\\ S_{\rho\lambda\sigma\tau}K^{\rho\lambda\sigma\tau} &\equiv \frac{1}{2}S_{\rho\lambda\sigma\tau}S^{\rho\lambda\sigma\tau} - \frac{15}{4}\mathbf{R}^2\mathbf{R}^2. \quad (32)\end{aligned}$$

Comparing the above two equations by referring to (28) , we found

$$S_{\alpha\lambda\sigma\tau}K_\beta{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\alpha\beta}S_{\rho\lambda\sigma\tau}K^{\rho\lambda\sigma\tau}. \quad (33)$$

Using (33) and considering the quadratic of $B_{\alpha\beta\mu\nu}$ in (26), we obtained

$$\begin{aligned} B_{\alpha\lambda\sigma\tau}B_\beta{}^{\lambda\sigma\tau} &\equiv -\frac{1}{4}S_{\alpha\lambda\sigma\tau}S_\beta{}^{\lambda\sigma\tau} + K_{\alpha\lambda\sigma\tau}K_\beta{}^{\lambda\sigma\tau} - \frac{15}{32}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2, \quad (34)\\ B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} &\equiv -\frac{1}{4}S_{\rho\lambda\sigma\tau}S^{\rho\lambda\sigma\tau} + K_{\rho\lambda\sigma\tau}K^{\rho\lambda\sigma\tau} - \frac{15}{8}\mathbf{R}^2\mathbf{R}^2. \quad (35)\end{aligned}$$

Comparing these two equations by using the identities (27) and (28), we found

$$K_{\alpha\lambda\sigma\tau}K_\beta{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\alpha\beta}K_{\rho\lambda\sigma\tau}K^{\rho\lambda\sigma\tau}. \quad (36)$$

Expand this identity explicitly:

$$\begin{aligned} K_{\alpha\lambda\sigma\tau}K_\beta{}^{\lambda\sigma\tau} &\equiv 2R_{\alpha\xi\lambda\kappa}R_\sigma{}^\xi{}_\tau{}^\kappa(R_{\beta\mu}{}^\lambda{}_\nu R^{\sigma\mu\tau\nu} + R_{\beta\mu}{}^\lambda{}_\nu R^{\tau\mu\sigma\nu}) + \frac{9}{16}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2, \quad (37)\\ K_{\rho\lambda\sigma\tau}K^{\rho\lambda\sigma\tau} &\equiv 2R_{\rho\xi\lambda\kappa}R_\sigma{}^\xi{}_\tau{}^\kappa(R^\rho{}_\mu{}^\lambda{}_\nu R^{\sigma\mu\tau\nu} + R^\rho{}_\mu{}^\lambda{}_\nu R^{\tau\mu\sigma\nu}) + \frac{9}{4}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2. \quad (38)\end{aligned}$$

Using the result in (36), the above two expressions can be simplified as

$$R_{\alpha\xi\lambda\kappa}R_\sigma{}^\xi{}_\tau{}^\kappa(R_{\beta\mu}{}^\lambda{}_\nu R^{\sigma\mu\tau\nu}+R_{\beta\mu}{}^\lambda{}_\nu R^{\tau\mu\sigma\nu}) \equiv \frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_\sigma{}^\xi{}_\tau{}^\kappa(R^\rho{}_\mu{}^\lambda{}_\nu R^{\sigma\mu\tau\nu}+R^\rho{}_\mu{}^\lambda{}_\nu R^{\tau\mu\sigma\nu}). \quad (39)$$

Rewriting (39) in an abbreviated notation,

$$\tilde{K}_{\alpha\lambda\sigma\tau}\tilde{K}_\beta{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\alpha\beta}\tilde{K}_{\rho\lambda\sigma\tau}\tilde{K}^{\rho\lambda\sigma\tau}. \quad (40)$$

Likewise, we found the quadratic $V_{\alpha\beta\mu\nu}$ identity to be

$$V_{\alpha\lambda\sigma\tau}V_{\beta}{}^{\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta}V_{\rho\lambda\sigma\tau}V^{\rho\lambda\sigma\tau}. \tag{41}$$

Moreover, we found some more identities in a similar way:

$$\begin{aligned} B_{\alpha\lambda\sigma\tau}K_{\beta}{}^{\lambda\sigma\tau} &\equiv \frac{1}{4}g_{\alpha\beta}B_{\rho\lambda\sigma\tau}K^{\rho\lambda\sigma\tau}, \qquad B_{\alpha\lambda\sigma\tau}V_{\beta}{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\alpha\beta}B_{\rho\lambda\sigma\tau}V^{\rho\lambda\sigma\tau}, &(42)\\ S_{\alpha\lambda\sigma\tau}V_{\beta}{}^{\lambda\sigma\tau} &\equiv \frac{1}{4}g_{\alpha\beta}S_{\rho\lambda\sigma\tau}V^{\rho\lambda\sigma\tau}, \qquad K_{\alpha\lambda\sigma\tau}V_{\beta}{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\alpha\beta}K_{\rho\lambda\sigma\tau}V^{\rho\lambda\sigma\tau}. &(43)\end{aligned}$$

We list all the results in a single formula as follows:

$$X_{\alpha\lambda\sigma\tau}Y_{\beta}{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\alpha\beta}X_{\rho\lambda\sigma\tau}Y^{\rho\lambda\sigma\tau}, \tag{44}$$

for all $X, Y \in \{B, S, K, V\}$. More fundamentally, we found

$$\tilde{X}_{\alpha\lambda\sigma\tau}\tilde{Y}_{\beta}{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\alpha\beta}\tilde{X}_{\rho\lambda\sigma\tau}\tilde{Y}^{\rho\lambda\sigma\tau}, \tag{45}$$

for all $\tilde{X}, \tilde{Y} \in \{\tilde{B}, \tilde{S}, \tilde{K}, \tilde{T}\}$. Bear in mind the symmetry of $\tilde{X}_{\alpha\beta\mu\nu} = \tilde{X}_{(\alpha\beta)(\mu\nu)} = \tilde{X}_{(\mu\nu)(\alpha\beta)}$. There comes a question whether the quadratic one-quarter identity which is shown in (45) requires some kind of symmetry property? The answer is no and we will discuss this in section 4.

4 Quadratic one-quarter metric identity

Expand (27) in an explicit form

$$\begin{aligned} B_{\alpha\lambda\sigma\tau}B_{\beta}{}^{\lambda\sigma\tau} &\equiv 2R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} + R_{\beta\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu}) - \frac{1}{16}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2, &(46)\\ B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} &\equiv 2R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} + R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu}) - \frac{1}{4}\mathbf{R}^2\mathbf{R}^2. &(47)\end{aligned}$$

Using the result in (27), we found the following relationship from (46) and (47):

$$\begin{aligned} &R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} + R_{\beta\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu})\\ \equiv\; &\frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} + R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu}). \qquad (48)\end{aligned}$$

There comes a natural question: whether the following are true independently,

$$\begin{aligned} R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} &\equiv \frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu}, &(49)\\ R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu} &\equiv \frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu}. &(50)\end{aligned}$$

Indeed they are, and this was verified by Edgar and Wingbrant [17]. Here we suggest another and perhaps an easier way to obtain this result and some other similar results.

The representations of the quadratic Bel-Robinson tensor are not unique, as is shown in (46) and (47). Because $B_{\alpha\beta\mu\nu}$ is completely symmetric, we found some more different expressions, including

$$B_{\alpha\lambda\sigma\tau}B_{\beta}{}^{\lambda\sigma\tau} \equiv 2R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R_{\beta\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu} + R_{\beta\mu}{}^{\sigma}{}_{\nu}R^{\tau\mu\lambda\nu}) \tag{51}$$

$$\equiv 2R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R_{\beta\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu} + R_{\beta\mu}{}^{\tau}{}_{\nu}R^{\lambda\mu\sigma\nu}) \tag{52}$$

$$\equiv 2R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R_{\beta\mu}{}^{\tau}{}_{\nu}R^{\lambda\mu\sigma\nu} + R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\tau\mu\sigma\nu}) - \frac{1}{32}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2 \tag{53}$$

$$\equiv 2R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} + R_{\beta\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu}) - \frac{1}{16}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2. \tag{54}$$

The corresponding contracted expressions are

$$B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} \equiv 2R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu} + R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\tau\mu\lambda\nu}) \tag{55}$$

$$\equiv 2R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu} + R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\lambda\mu\sigma\nu}) \tag{56}$$

$$\equiv 2R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\lambda\mu\sigma\nu} + R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\tau\mu\sigma\nu}) - \frac{1}{8}\mathbf{R}^2\mathbf{R}^2 \tag{57}$$

$$\equiv 2R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} + R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu}) - \frac{1}{4}\mathbf{R}^2\mathbf{R}^2. \tag{58}$$

Examining the four pairs of equations (51) and (52), (55) and (56), (52) and (53), (56) and (57), we found

$$R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\mu}{}^{\sigma}{}_{\nu}R^{\tau\mu\lambda\nu} \equiv R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\mu}{}^{\tau}{}_{\nu}R^{\lambda\mu\sigma\nu}, \tag{59}$$

$$R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\tau\mu\lambda\nu} \equiv R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\lambda\mu\sigma\nu}, \tag{60}$$

$$R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu} \equiv R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\tau\mu\sigma\nu} - \frac{1}{32}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2, \tag{61}$$

$$R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu} \equiv R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\tau\mu\sigma\nu} - \frac{1}{8}\mathbf{R}^2\mathbf{R}^2. \tag{62}$$

Note that (59) and (60) are trivial equalities, because it can be obtained from renaming the dummy indices.

Here we list out the two explicit quadratic expressions of $S_{\alpha\beta\mu\nu}$:

$$S_{\alpha\lambda\sigma\tau}S_{\beta}{}^{\lambda\sigma\tau} \equiv 2R_{\alpha\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}(R_{\beta}{}^{\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu} + R_{\beta}{}^{\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu}) + \frac{1}{2}g_{\alpha\beta}\mathbf{R}^2\mathbf{R}^2, \tag{63}$$

$$S_{\rho\lambda\sigma\tau}S^{\rho\lambda\sigma\tau} \equiv 2R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}(R^{\rho\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu} + R^{\rho\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu}) + 2\mathbf{R}^2\mathbf{R}^2. \tag{64}$$

Using (28), the relationship between (63) and (64) becomes

$$R_{\alpha\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}(R_{\beta}{}^{\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu} + R_{\beta}{}^{\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu})$$

$$\equiv \frac{1}{4}g_{\alpha\beta}R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}(R^{\rho\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu} + R^{\rho\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu}). \tag{65}$$

As before, like the situation of Edgar and Wingbrant [17], one may wonder whether the following are true independently for any frames

$$R_{\alpha\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R_{\beta}{}^{\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu} \equiv \frac{1}{4}g_{\alpha\beta}R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu}, \tag{66}$$

$$R_{\alpha\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R_{\beta}{}^{\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu} \equiv \frac{1}{4}g_{\alpha\beta}R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu}. \tag{67}$$

Once again, we found they are indeed true, having verified these relations in orthonormal frames. Moreover, using symmetry properties, we also obtained

$$R_{\alpha\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R_{\beta}{}^{\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu} \equiv R_{\alpha\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R_{\beta}{}^{\sigma}{}_{\mu\nu}R^{\lambda\tau\mu\nu}, \tag{68}$$

$$R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu} \equiv R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\sigma}{}_{\mu\nu}R^{\lambda\tau\mu\nu}. \tag{69}$$

Based on the first Bianchi identity, we found the following relations:

$$R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} - R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\tau\mu\sigma\nu}) \equiv \frac{1}{8}R_{\alpha\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R_{\beta}{}^{\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu}, \tag{70}$$

$$R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} - R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\tau\mu\sigma\nu}) \equiv \frac{1}{8}R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu}. \tag{71}$$

Referring to (39), (66), (70) and (71), we found

$$R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} \equiv \frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu}, \tag{72}$$

$$R_{\alpha\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R_{\beta\mu}{}^{\lambda}{}_{\nu}R^{\tau\mu\sigma\nu} \equiv \frac{1}{4}g_{\alpha\beta}R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\tau\mu\sigma\nu}. \tag{73}$$

Consider the difference of the two terms on the right hand side of (55)

$$\begin{aligned} & R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu} - R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\tau\mu\lambda\nu} \\ \equiv\ & \frac{1}{16}(2R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu} + 4R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu} - \mathbf{R}^2\mathbf{R}^2) \equiv 0, \end{aligned} \tag{74}$$

where we have found and made use of the following identity, which was verified by using orthonormal frames:

$$2R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu} + 4R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu} \equiv \mathbf{R}^2\mathbf{R}^2. \tag{75}$$

Using the result in (74), we noted that (55) can be rewritten as

$$\frac{1}{4}B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} \equiv R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu} \equiv R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\tau\mu\lambda\nu}. \tag{76}$$

Using (76), refer to (57), we discovered

$$\frac{1}{4}B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} \equiv R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\tau\mu\sigma\nu} - \frac{1}{16}\mathbf{R}^2\mathbf{R}^2. \tag{77}$$

In order to take care of the first two terms on the right hand side of (58) in a similar way, we need to use another identity which comes from Deser [18]

$$2B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} \equiv \mathbf{R}^2\mathbf{R}^2 - 2R_{\rho\lambda\xi\kappa}R_{\sigma\tau}{}^{\xi\kappa}R^{\rho\lambda}{}_{\mu\nu}R^{\sigma\tau\mu\nu}. \tag{78}$$

We have checked the above identity using orthonormal frames, however, our result presented here differs from [18] by the coefficient of '2' at the last term in (78). According to (58) and substituting this amazing identity in (78), we obtained

$$R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\lambda}{}_{\nu}R^{\sigma\mu\tau\nu} \equiv \frac{1}{8}B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} + \frac{1}{8}\mathbf{R}^2\mathbf{R}^2, \tag{79}$$

$$R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu} \equiv \frac{3}{8}B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau}. \tag{80}$$

There are only four fourth rank quadratic-in-Weyl-curvature tensors having the specified symmetry; they were mentioned previously: (21) to (24). From this it follows that there exits only two independent quadratic-in-Weyl-curavture scalar expressions. Explicitly

$$R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu}, \quad \mathbf{R}^2\mathbf{R^2}, \tag{81}$$

(for this result see also [18]). The others are just the linear combinations of these two. For example

$$\begin{aligned} R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho\tau}{}_{\mu\nu}R^{\sigma\lambda\mu\nu} &= 2R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}(R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\sigma\mu\lambda\nu} - R^{\rho}{}_{\mu}{}^{\tau}{}_{\nu}R^{\lambda\mu\sigma\nu}) \\ &= R_{\rho\xi\lambda\kappa}R_{\sigma}{}^{\xi}{}_{\tau}{}^{\kappa}R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}R^{\lambda\mu\tau\nu}. \end{aligned} \tag{82}$$

After going through some messy algebra, the two basis components for the square quadratic curvature tensors which were mentioned in (81) denote a simple fact. There exists an identity (i.e., not a condition) such that

$$X_{\alpha\lambda\sigma\tau}Y_{\beta}{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\alpha\beta}X_{\rho\lambda\sigma\tau}Y^{\rho\lambda\sigma\tau}, \tag{83}$$

where $X_{\alpha\lambda\sigma\tau}$ and $Y_{\beta\lambda\sigma\tau}$ are any tensors quadratic in the Riemann curvature, non-vanishing in vacuum.

Looking back at the identity in (78), there comes a completeness question. As the Bel-Robinson tensor satisfies the dominant energy condition, which means the sign of $B_{\alpha\beta\mu\nu}u^{\alpha}v^{\beta}w^{\mu}z^{\nu}$ is non-negative. Does there exist a definite sign of the quadratic Bel-Robinson tensor? Checking the sign of $B^2_{\alpha\beta\mu\nu}$, we used the five distinct Petrov types [19] as a verification technique in orthonormal frames, we found

$$B_{\alpha\beta\mu\nu}B^{\alpha\beta\mu\nu} \geq 0. \tag{84}$$

This result indicates that it is true for all frames because $B_{\alpha\beta\mu\nu}$ is a tensor. Alternatively, using the $(3+1)$ decomposition and the identity in (75), we recovered the same result

$$\begin{aligned} B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} &= \frac{1}{4}\mathbf{R}^2\mathbf{R}^2 + \frac{2}{3}(R_{0a\xi\kappa}R_{bc}{}^{\xi\kappa} + R_{0b\xi\kappa}R_{ca}{}^{\xi\kappa} + R_{0c\xi\kappa}R_{ab}{}^{\xi\kappa}) \\ &\quad \times(R_{0}{}^{a}{}_{\mu\nu}R^{bc\mu\nu} + R_{0}{}^{b}{}_{\mu\nu}R^{ca\mu\nu} + R_{0}{}^{c}{}_{\mu\nu}R^{ab\mu\nu}) \geq 0. \end{aligned} \tag{85}$$

For the completeness, as $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ share the same value of the energy-momentum density in vacuum, one may wonder what is the sign of the quadratic of $V_{\alpha\beta\mu\nu}$? We found that, unfortunately, the sign is not certain. Here is the simple derivation

$$\begin{aligned} V_{\rho\lambda\sigma\tau}V^{\rho\lambda\sigma\tau} &= (B_{\rho\lambda\sigma\tau} + W_{\rho\lambda\sigma\tau})(B^{\rho\lambda\sigma\tau} + W^{\rho\lambda\sigma\tau}) \\ &= B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau} + 2B_{\rho\lambda\sigma\tau}W^{\rho\lambda\sigma\tau} + W_{\rho\lambda\sigma\tau}W^{\rho\lambda\sigma\tau} \\ &= \frac{1}{8}(9\mathbf{R}^2\mathbf{R}^2 - 10B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau}). \end{aligned} \tag{86}$$

where

$$B_{\rho\lambda\sigma\tau}W^{\rho\lambda\sigma\tau} = 0, \quad W_{\rho\lambda\sigma\tau}W^{\rho\lambda\sigma\tau} = \frac{9}{8}(\mathbf{R}^2\mathbf{R}^2 - 2B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau}). \tag{87}$$

In Petrov type II [19], (86) becomes

$$V_{\rho\lambda\sigma\tau}V^{\rho\lambda\sigma\tau} = 9[13(E_{11}^2 - H_{11}^2)^2 - 20E_{11}^2H_{11}^2]. \tag{88}$$

Note that if either E_{11} is much larger than H_{11} or conversely, $V_{\rho\lambda\sigma\tau}V^{\rho\sigma\tau}$ is positive. However, if E_{11} is very close to H_{11}, then the sign of $V_{\rho\lambda\sigma\tau}V^{\rho\lambda\sigma\tau}$ will become negative. Hence the sign of $V_{\rho\lambda\sigma\tau}V^{\rho\lambda\sigma\tau}$ is not certain.

5 Algebraic Rainich conditions

The original algebraic Rainich conditions use the Ricci tensor. Making an analogy from the second rank to a fourth rank traceless tensor, we write

$$X_{\alpha\lambda\sigma\tau}X_{\beta}{}^{\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta}X_{\rho\lambda\sigma\tau}X^{\rho\lambda\sigma\tau}, \quad 0 = X^{\alpha}{}_{\alpha\mu\nu} = X^{\alpha}{}_{\mu\alpha\nu} = ..., \quad X_{\alpha\beta\mu\nu}u^{\alpha}u^{\beta}u^{\mu}u^{\nu} \geq 0, \tag{89}$$

where u is timelike unit normal vector (the latter relation is equivalent to $X_{0000} \geq 0$). Note that we assumed the coefficient of $X_{\alpha\beta\mu\nu}$ is positive. The basic idea of the algebraic Rainich conditions do not require the dominant energy condition, but just the weak energy condition [9]. For the fourth rank tensor, as far as the quasilocal in the small sphere limit is concerned, we found that only $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ satisfy these algebraic Rainich conditions. Moreover, we modify the algebraic Rainich conditions as follows:

$$0 = X^{\alpha}{}_{\alpha\mu\nu} = X^{\alpha}{}_{\mu\alpha\nu} = ..., \quad X_{\alpha\beta\mu\nu}u^{\alpha}u^{\beta}u^{\mu}u^{\nu} \geq 0, \tag{90}$$

where the first requirement in (89) is ignored because it is an identity but not a condition, which is explained in section 4. Here we found that the completely traceless property of $X_{\alpha\beta\mu\nu}$ gives the basis of $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$, and these two tensors imply positivity (more precisely inside the forward light cone, briefly causal). Interestingly, this is also true conversely. Therefore the fourth rank algebraic Rainich conditions can be further simplified. In short

$$0 = X^{\alpha}{}_{\alpha\mu\nu} = X^{\alpha}{}_{\mu\alpha\nu} = ... \quad \Leftrightarrow \quad X_{\alpha\beta\mu\nu}u^{\beta}u^{\mu}u^{\nu} = (E_{ab}E^{ab} + H_{ab}H^{ab}, 2\epsilon_{cab}E^{ad}H^{b}{}_{d}). \tag{91}$$

This indicates that, as far as the quasilocal small sphere limit, the algebraic Rainich conditions only require one condition. In other words, either the completely traceless or positivity (i.e., causal) is sufficient. The following is the simple proof.

Case (i). Completely traceless property implies positivity (more precisely causal). Recall the four basic tensors from (21) to (24), because of the symmetries of $\tilde{B}_{\alpha\beta\mu\nu}$, $\tilde{S}_{\alpha\beta\mu\nu}$, $\tilde{K}_{\alpha\beta\mu\nu}$ and $\tilde{T}_{\alpha\beta\mu\nu}$, consider the two following totally traceless statements:

$$0 = a_1\tilde{B}^{\alpha}{}_{\alpha\mu\nu} + a_2\tilde{S}^{\alpha}{}_{\alpha\mu\nu} + a_3\tilde{K}^{\alpha}{}_{\alpha\mu\nu} + a_4\tilde{T}^{\alpha}{}_{\alpha\mu\nu} = \frac{1}{2}(a_1 + a_2 - a_4)g_{\mu\nu}\mathbf{R}^2, \tag{92}$$

$$0 = a_1\tilde{B}^{\alpha}{}_{\mu\alpha\nu} + a_2\tilde{S}^{\alpha}{}_{\mu\alpha\nu} + a_3\tilde{K}^{\alpha}{}_{\mu\alpha\nu} + a_4\tilde{T}^{\alpha}{}_{\mu\alpha\nu} = \frac{1}{8}(a_1 - 2a_2 + 3a_3 - a_4)g_{\mu\nu}\mathbf{R}^2. \tag{93}$$

Then we have two constraints,

$$0 = a_1 + a_2 - a_4, \tag{94}$$
$$0 = a_1 - 2a_2 + 3a_3 - a_4. \tag{95}$$

The solution for the above two equations can be represented as

$$a_4 = a_1 + a_2, \quad a_2 = a_3. \tag{96}$$

Then the general linear combination of the four basic fundamental tensors (i.e., confined in the quasilocal small region) can be reduced as

$$\begin{aligned} & a_1\tilde{B}_{\alpha\beta\mu\nu} + a_2\tilde{S}_{\alpha\beta\mu\nu} + a_3\tilde{K}_{\alpha\beta\mu\nu} + a_4\tilde{T}_{\alpha\beta\mu\nu} \\ = \ & a_1(\tilde{B}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu\nu}) + a_2(\tilde{S}_{\alpha\beta\mu\nu} + \tilde{K}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu\nu}) \\ = \ & a_1 B_{\alpha\beta\mu\nu} + a_2 V_{\alpha\beta\mu\nu}. \end{aligned} \tag{97}$$

This result indicates that there are only two tensors $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ which satisfy the completely trace free property and form a linear basis. Obviously, they also fulfill the positivity (i.e., causal)

$$B_{\mu 000} = V_{\mu 000} = (E_{ab}E^{ab} + H_{ab}H^{ab},\ 2\epsilon_{cab}E^{ad}H^{bd}), \tag{98}$$

where the energy and momentum density represent the causal relationship:

$$E_{ab}E^{ab} + H_{ab}H^{ab} \geq |2\epsilon_{cab}E^{ad}H^{b}{}_{d}| \geq 0. \tag{99}$$

Case (ii). Positivity (more precisely causal) implies completely traceless. First of all, why do we keep emphasizing the positivity and causal? The reason is that positivity alone cannot imply completely trace free. In particular, suppose

$$X_{\alpha\beta\mu\nu} = R_{\alpha\lambda\beta\sigma}R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma}. \tag{100}$$

Although $X_{\alpha\beta\mu\nu}$ preserves the positive condition, it does not satisfy the completely traceless property. Explicitly

$$X^{\alpha}{}_{\mu\alpha\nu} = \frac{1}{4}g_{\mu\nu}\mathbf{R}^2 \neq 0, \quad X_{0000} = E_{ab}E^{ab} \geq 0. \tag{101}$$

Returning back to causal, consider the energy-momentum integral in a quasilocal small sphere with constant time evolution of the hypersurface. Note that the fourth rank tensor $X_{\alpha\beta\mu\nu}$ needs to be symmetric at the last two indices because of the small sphere limit

$$\begin{aligned} N^{\mu}P_{\mu} &= \int N^{\mu}X^{\rho}{}_{\mu\xi\kappa}x^{\xi}x^{\kappa}\eta_{\rho} = \int N^{\mu}X^{0}{}_{\mu\xi\kappa}x^{\xi}x^{\kappa}\eta_{0} = \int N^{\mu}X^{0}{}_{\mu ij}x^{i}x^{j}dV \\ &= \int N^{\mu}X^{0}{}_{\mu l}{}^{l}\frac{r^2}{3}dV = \int N^{\mu}X^{0}{}_{\mu l}{}^{l}\frac{r^2}{3}4\pi r^2 dr = N^{\mu}X^{0}{}_{\mu l}{}^{l}\frac{4\pi r^5}{15} \\ &= N^{\mu}\left(X^{0}{}_{\mu\alpha}{}^{\alpha} - X^{0}{}_{\mu 0}{}^{0}\right)\frac{4\pi r^5}{15} = N^{\mu}X^{0}{}_{\mu 00}\frac{4\pi r^5}{15}, \end{aligned} \tag{102}$$

where we made the assumption that $X_{0\mu\alpha}{}^{\alpha}$ vanishes and fulfills causal (i.e., Lorentz-covariant, see section 4.2.2 of [20]). Consider now the requirement for the energy-momentum being future pointing and non-spacelike (i.e., causal) in the small sphere limit :

$$\begin{aligned}
& a_1\tilde{B}_{\mu 0l}{}^{l} + a_2\tilde{S}_{\mu 0l}{}^{l} + a_3\tilde{K}_{\mu 0l}{}^{l} + a_4\tilde{T}_{\mu 0l}{}^{l} \\
= \; & a_1(-2E_{ab}E^{ab} + 4H_{ab}H^{ab},\ 2\epsilon_{cab}E^{ad}H^{b}{}_{d}) + a_2(-4E_{ab}E^{ab} + 4H_{ab}H^{ab},\ 0) \\
& +a_3(2E_{ab}E^{ab},\ 2\epsilon_{cab}E^{ad}H^{b}{}_{d}) + a_4(3E_{ab}E^{ab} - 3H_{ab}H^{ab},\ 0) \\
= \; & (-2a_1 - 4a_2 + 2a_3 + 3a_4)E_{ab}E^{ab} + (4a_1 + 4a_2 - 3a_4)H_{ab}H^{ab} \\
& +(2a_1 + 2a_3)\epsilon_{cab}E^{ad}H^{b}{}_{d}. \qquad (103)
\end{aligned}$$

Causal (i.e., Lorentz-covariant, see [20]) requires the magnitude of $E_{ab}E^{ab}$ and $H_{ab}H^{ab}$ to be the same and the energy is greater than or equal to the momentum as shown in (99). Simply, we need to calculate two equations from (103), but it turns out that one constraint is enough, requiring the coefficients of the electric and magnetic square parts to be the same

$$-2a_1 - 4a_2 + 2a_3 + 3a_4 = 4a_1 + 4a_2 - 3a_4, \qquad (104)$$

Then a_4 can be written in terms of a_1, a_2 and a_3

$$a_4 = a_1 + \frac{4a_2}{3} - \frac{a_3}{3}. \qquad (105)$$

Substituting (105) into (103), we found

$$\begin{aligned}
& a_1\tilde{B}_{\mu 0l}{}^{l} + a_2\tilde{S}_{\mu 0l}{}^{l} + a_3\tilde{K}_{\mu 0l}{}^{l} + a_4\tilde{T}_{\mu 0l}{}^{l} \\
= \; & a_1(\tilde{B}_{\mu 0l}{}^{l} + \tilde{T}_{\mu 0l}{}^{l}) + a_2\left(\tilde{S}_{\mu 0l}{}^{l} + \frac{4}{3}\tilde{T}_{\mu 0l}{}^{l}\right) + a_3\left(\tilde{K}_{\mu 0l}{}^{l} - \frac{1}{3}\tilde{T}_{\mu 0l}{}^{l}\right) \\
= \; & a_1 B_{\mu 0l}{}^{l} + a_3\left(\tilde{K}_{\mu 000} - \frac{1}{3}\tilde{T}_{\mu 0l}{}^{l}\right) \\
= \; & a_1 B_{\mu 000} + a_3(\tilde{K}_{\mu 000} + \tilde{S}_{\mu 000} + \tilde{T}_{\mu 000}) \\
= \; & a_1 B_{\mu 000} + a_3 V_{\mu 000} \\
= \; & (a_1 + a_3)B_{\mu 000}, \qquad (106)
\end{aligned}$$

where we require $a_1 + a_3 \geq 0$ and made the following substitutions

$$\begin{aligned}
\tilde{S}_{\mu 0l}{}^{l} &= -\frac{4}{3}\tilde{T}_{\mu 0l}{}^{l}, && (107) \\
B_{\mu 0\alpha}{}^{\alpha} &= 0 = \tilde{K}_{\mu 0\alpha}{}^{\alpha}, && (108) \\
B_{\alpha\beta\mu\nu} &= \tilde{B}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu\nu}, && (109) \\
V_{\alpha\beta\mu\nu} &= \tilde{S}_{\alpha\beta\mu\nu} + \tilde{K}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu\nu}, && (110)
\end{aligned}$$

and we also used

$$-\frac{1}{3}\tilde{T}_{\mu 0l}{}^{l} = \tilde{S}_{\mu 000} + \tilde{T}_{\mu 000}, \qquad (111)$$

which can easily be verified.

Hence, the completely traceless and causal properties form necessary and sufficient conditions. This means we can further simplify the algebraic Rainich conditions for a fourth rank tensor; as far as the quasilocal small sphere limit is concerned, we only need the completely trace free condition or positivity (i.e., causal). This is an interesting result which is valid in the quasilocal small sphere region.

Moreover, we have found another interesting result for the general case (i.e., not confined to the quasilocal small sphere limit) which will be discussed in the next section.

6 Algebraic Rainich conditions for general fourth rank tensor

For the fourth rank tensors, so far we only confined ourselves in the quasilocal small sphere limit, $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$. What about the general situation for this fourth rank tensor? Generally speaking, we found remarkably that the algebraic Rainich conditions only require the completely trace free property. This means the totally traceless condition automatically fulfills the positivity; however, the converse does not apply.

In principle, assuming vacuum (i.e., the Ricci tensor vanishes), using $R_{\alpha\beta\mu\nu} = R_{[\alpha\beta][\mu\nu]} = R_{\mu\nu\alpha\beta}$ we can get the eighteen combinations for the fourth rank quadratic Riemann curvature tensors:

$R_{\alpha\lambda\mu\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma}$	$R_{\alpha\mu\lambda\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma}$	$R_{\alpha\lambda\mu\sigma}R_{\beta\nu}{}^{\lambda\sigma}$	$R_{\alpha\mu\lambda\sigma}R_{\beta\nu}{}^{\lambda\sigma}$	$R_{\alpha\lambda\nu\sigma}R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma}$
$R_{\alpha\nu\lambda\sigma}R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma}$	$R_{\alpha\lambda\nu\sigma}R_{\beta\mu}{}^{\lambda\sigma}$	$R_{\alpha\nu\lambda\sigma}R_{\beta\mu}{}^{\lambda\sigma}$	$R_{\alpha\lambda\mu\sigma}R_{\nu}{}^{\lambda}{}_{\beta}{}^{\sigma}$	$R_{\alpha\lambda\nu\sigma}R_{\mu}{}^{\lambda}{}_{\beta}{}^{\sigma}$
$R_{\alpha\lambda\beta\sigma}R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma}$	$R_{\alpha\beta\lambda\sigma}R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma}$	$R_{\alpha\lambda\beta\sigma}R_{\mu\nu}{}^{\lambda\sigma}$	$R_{\alpha\beta\lambda\sigma}R_{\mu\nu}{}^{\lambda\sigma}$	$R_{\alpha\lambda\beta\sigma}R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma}$
$g_{\alpha\beta}g_{\mu\nu}R^2$	$g_{\alpha\mu}g_{\beta\nu}R^2$	$g_{\alpha\nu}g_{\beta\mu}R^2$		

Table 1: Eighteen quadratic Riemann curvature tensors

From the first Bianich identity $R_{\alpha[\beta\mu\nu]} = 0$ and the identity (25), these can be reduced to the following eight algebraically linearly independent expressions:

$$\begin{array}{llll} R_{\alpha\lambda\mu\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma}, & R_{\alpha\lambda\nu\sigma}R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma}, & R_{\alpha\mu\lambda\sigma}R_{\beta\nu}{}^{\lambda\sigma}, & R_{\alpha\nu\lambda\sigma}R_{\beta\mu}{}^{\lambda\sigma}, \\ R_{\alpha\lambda\beta\sigma}R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma}, & R_{\alpha\lambda\beta\sigma}R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma}, & g_{\alpha\mu}g_{\beta\nu}R^2, & g_{\alpha\nu}g_{\beta\mu}R^2. \end{array} \tag{112}$$

Here we consider two cases.

Case (i). Completely trace free, without any symmetry requirement. We define

$$\begin{aligned} X_{\alpha\beta\mu\nu} \;:=\; & b_1 R_{\alpha\lambda\mu\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma} + b_2 R_{\alpha\lambda\nu\sigma}R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma} + b_3 R_{\alpha\mu\lambda\sigma}R_{\beta\nu}{}^{\lambda\sigma} + b_4 R_{\alpha\nu\lambda\sigma}R_{\beta\mu}{}^{\lambda\sigma} \\ & + b_5 R_{\alpha\lambda\beta\sigma}R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma} + b_6 R_{\alpha\lambda\beta\sigma}R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma} + b_7 g_{\alpha\mu}g_{\beta\nu}R^2 + b_8 g_{\alpha\nu}g_{\beta\mu}R^2, \end{aligned} \tag{113}$$

where b_1 to b_8 are constants. Basically, in order to set the totally traceless for $X_{\alpha\beta\mu\nu}$, there should be six possible combinations. However, we found there are only three

independent constraints. Explicitly

$$0 = X^{\alpha}{}_{\alpha\mu\nu} = X_{\mu\nu}{}^{\alpha}{}_{\alpha} = \left(\frac{b_1}{4} + \frac{b_2}{4} + \frac{b_3}{4} + \frac{b_4}{4} + b_7 + b_8\right) g_{\mu\nu}\mathbf{R}^2, \tag{114}$$

$$0 = X^{\alpha}{}_{\mu\alpha\nu} = X_{\mu}{}^{\alpha}{}_{\nu\alpha} = \left(\frac{b_2}{8} - \frac{b_4}{4} + \frac{b_5}{4} + \frac{b_6}{8} + 4b_7 + b_8\right) g_{\mu\nu}\mathbf{R}^2, \tag{115}$$

$$0 = X^{\alpha}{}_{\mu\nu\alpha} = X_{\mu}{}^{\alpha}{}_{\alpha\nu} = \left(\frac{b_1}{8} - \frac{b_3}{4} + \frac{b_5}{8} + \frac{b_6}{4} + b_7 + 4b_8\right) g_{\mu\nu}\mathbf{R}^2. \tag{116}$$

Using the constraints from (114) to (116), we eliminate b_6, b_7, b_8 and rewrite (113) as

$$\begin{aligned} X_{\alpha\beta\mu\nu} \;=\;& b_1\left(R_{\alpha\lambda\mu\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma} + 3R_{\alpha\lambda\beta\sigma}B_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma} - \frac{1}{24}g_{\alpha\mu}g_{\beta\nu}\mathbf{R}^2 - \frac{5}{24}g_{\alpha\nu}g_{\beta\mu}\mathbf{R}^2\right) \\ &+b_2\left(R_{\alpha\lambda\nu\sigma}R_{\beta}{}^{\lambda}{}_{\mu}{}^{\sigma} + 3R_{\alpha\lambda\beta\sigma}R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma} - \frac{1}{12}g_{\alpha\mu}g_{\beta\nu}\mathbf{R}^2 - \frac{1}{6}g_{\alpha\nu}g_{\beta\mu}\mathbf{R}^2\right) \\ &+b_3\left(R_{\alpha\mu\lambda\sigma}R_{\beta\nu}{}^{\lambda\sigma} + 4R_{\alpha\lambda\beta\sigma}R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma} - \frac{1}{12}g_{\alpha\mu}g_{\beta\nu}\mathbf{R}^2 - \frac{1}{6}g_{\alpha\nu}g_{\beta\mu}\mathbf{R}^2\right) \\ &+b_4\left(R_{\alpha\nu\lambda\sigma}R_{\beta\mu}{}^{\lambda\sigma} + 4R_{\alpha\lambda\beta\sigma}R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma} - \frac{1}{4}g_{\alpha\nu}g_{\beta\mu}\mathbf{R}^2\right) \\ &+b_5\left(R_{\alpha\lambda\beta\sigma}R_{\mu}{}^{\lambda}{}_{\nu}{}^{\sigma} - R_{\alpha\lambda\beta\sigma}R_{\nu}{}^{\lambda}{}_{\mu}{}^{\sigma} - \frac{1}{24}g_{\alpha\mu}g_{\beta\nu}\mathbf{R}^2 + \frac{1}{24}g_{\alpha\nu}g_{\beta\mu}\mathbf{R}^2\right). \end{aligned} \tag{117}$$

Note that in general this is not simply a linear combination of $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$. We found that the totally traceless property implies positivity,

$$0 = X^{\alpha}{}_{\alpha\mu\nu} = X^{\alpha}{}_{\mu\alpha\nu} = ... \quad \Rightarrow \quad X_{0000} \geq 0. \tag{118}$$

as long as b_1 to b_5 are all non-negative. But the converse is not true in general. In particular

$$X_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma}R_{\beta}{}^{\lambda}{}_{\nu}{}^{\sigma}, \tag{119}$$

satisfies the positivity requirement

$$X_{0000} = E_{ab}E^{ab} \geq 0. \tag{120}$$

But does not satisfy the totally traceless property, since

$$X^{\alpha}{}_{\alpha\mu\nu} = R^{\alpha}{}_{\lambda\mu\sigma}R_{\alpha}{}^{\lambda}{}_{\nu}{}^{\sigma} = \frac{1}{4}g_{\mu\nu}\mathbf{R}^2 \neq 0. \tag{121}$$

Case (ii). Completely trace free with symmetry requirement. We impose one symmetry condition and then allow the totally trace free condition afterward. Set $X'_{\alpha\beta\mu\nu} = X'_{\alpha\beta(\mu\nu)}$ and it will have the following implications simultaneously

$$X'_{\alpha\beta\mu\nu} = X'_{\alpha\beta(\mu\nu)} \quad \Rightarrow \quad X'_{\alpha\beta\mu\nu} = X'_{(\alpha\beta)\mu\nu} \quad \text{and} \quad X'_{\alpha\beta\mu\nu} = X'_{\mu\nu\alpha\beta}. \tag{122}$$

For instance, let

$$\begin{aligned}
X'_{\alpha\beta\mu\nu} &= c_1(R_{\alpha\lambda\mu\sigma}R_\beta{}^\lambda{}_\nu{}^\sigma + R_{\alpha\lambda\nu\sigma}R_\beta{}^\lambda{}_\mu{}^\sigma) + c_2(R_{\alpha\mu\lambda\sigma}R_{\beta\nu}{}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma}R_{\beta\mu}{}^{\lambda\sigma}) \\
&\quad + c_3(R_{\alpha\lambda\beta\sigma}R_\mu{}^\lambda{}_\nu{}^\sigma + R_{\alpha\lambda\beta\sigma}R_\nu{}^\lambda{}_\mu{}^\sigma) + c_4(g_{\alpha\mu}g_{\beta\nu}\mathbf{R}^2 + g_{\alpha\nu}g_{\beta\mu}\mathbf{R}^2) \\
&= (c_1 - 8c_4)\tilde{B}_{\alpha\beta\mu\nu} + (c_2 - 4c_4)\tilde{S}_{\alpha\beta\mu\nu} + (c_3 + 8c_4)\tilde{K}_{\alpha\beta\mu\nu} - 16c_4\tilde{T}_{\alpha\beta\mu\nu} \\
&= c'_1\tilde{B}_{\alpha\beta\mu\nu} + c'_2\tilde{S}_{\alpha\beta\mu\nu} + c'_3\tilde{K}_{\alpha\beta\mu\nu} + c'_4\tilde{T}_{\alpha\beta\mu\nu}, \qquad (123)
\end{aligned}$$

where c_1 to c_4 or c'_1 to c'_4 are constants, and we have made use of the property (25). In order to fulfill the completely trace free requirements, there are only two different constraints we need to consider

$$0 = X'^\alpha{}_{\alpha\mu\nu} = c'_1 + c'_2 - c'_4, \qquad (124)$$

$$0 = X'^\alpha{}_{\mu\alpha\nu} = c'_1 - 2c'_2 + 3c'_3 - c'_4. \qquad (125)$$

The solution for the above two equations are

$$c'_4 = c'_1 + c'_2, \quad c'_2 = c'_3. \qquad (126)$$

Using (126), rewrite (123) as

$$\begin{aligned}
X'_{\alpha\beta\mu\nu} &= c'_1(\tilde{B}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu\nu}) + c'_2(\tilde{S}_{\alpha\beta\mu\nu} + \tilde{K}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu\nu}) \\
&= c'_1B_{\alpha\beta\mu\nu} + c'_2V_{\alpha\beta\mu\nu}. \qquad (127)
\end{aligned}$$

Hence, starting from the general completely trace free property, we have recovered the unique basis $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ in the quasilocal small sphere limit.

This means that if we impose some certain symmetry as indicated in (122), we obtained the same result as mentioned in section 5 but without the quasilocal small sphere limit restriction. Explicitly, in general, the purely mathematical property (i.e., completely trace free) guarantees the physical requirements [7] (i.e., energy-momentum conservation and causal).

Likewise, for the completeness, we get the same result if one sets $X''_{\alpha\beta\mu\nu} = X''_{\alpha(\beta\mu)\nu}$.

7 Conclusion

The Bel-Robinson tensor satisfies the one-quarter quadratic identity $B_{\alpha\lambda\sigma\tau}B_\beta{}^{\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta}\ B_{\rho\lambda\sigma\tau}B^{\rho\lambda\sigma\tau}$. We found that the tensors $S_{\alpha\beta\mu\nu}$, $K_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ also satisfy the same interesting one-quarter quadratic identity as $B_{\alpha\beta\mu\nu}$ does. Explicitly $X_{\alpha\lambda\sigma\tau}Y_\beta{}^{\lambda\sigma\tau} \equiv \frac{1}{4}g_{\alpha\beta}X_{\rho\lambda\sigma\tau}Y^{\rho\lambda\sigma\tau}$, for all $X, Y \in \{B, S, K, V\}$. More fundamentally, for any quadratic Riemann curvature tensors $\tilde{X}_{\alpha\beta\mu\nu}$ and $\tilde{Y}_{\alpha\beta\mu\nu}$, we have the same result $\tilde{X}_{\alpha\lambda\sigma\tau}\tilde{Y}_\beta{}^{\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta}\tilde{X}_{\rho\lambda\sigma\tau}\tilde{Y}^{\rho\lambda\sigma\tau}$. This indicates that this is an identity and no longer a condition. Therefore the algebraic Rainich conditions left two conditions, not the original three.

Moreover, under the quasilocal small sphere limit restriction, we found that there are only two fourth rank tensors $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ forming a basis for good expressions. Both of them have the completely trace free and causal properties, these two form

necessary and sufficient conditions. Surprisingly, either completely traceless or causal can fulfill the algebraic Rainich conditions.

Furthermore, relaxing the quasilocal small sphere limit restriction and considering the general fourth rank tensor, we found two remarkable results. One is without any symmetry requirement: the algebraic conditions only require totally trace free. The other is imposing some certain symmetry: we recovered the same result as in the quasilocal small sphere limit (i.e., $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$).

Acknowledgment

This work was supported by NSC 97-2811-M-032-007 and NSC 98-2811-M-008-078.

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